Linear Algebra

[KOMS120301] - 2023/2024

14.1 - Inner Product Space and Orthogonality

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Learning objectives

After this lecture, you should be able to:

1. explain the concept of inner product in general (especially, in terms of 4 axioms of inner product);
2. explain the concept of inner product space;
3. compute the inner product of two vectors in a space;
4. compute the weighted inner product of two vectors;
5. compute the angle of two vectors in terms of inner product;
6. compute the distance of two vectors;
7. show that two vectors are orthogonal or not.

Part 1: Inner product & inner product space

Inner product

An inner product on a real vector space* V is a function that associates real numbers u, v with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors u, v, and w in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetric axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{w} \rangle$ **[Homogenity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a real inner product space.

*A real vector space is a vector space whose field of scalars is the field of reals.

Euclidean inner product

The inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is called Euclidean inner product (standard inner product).

Inner product vs dot product:

Algebraic properties of inner product space

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in a real inner product space V, and if $k \in \mathbb{R}$, then:

1. $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v} + \mathbf{0} \rangle = 0$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u} + \mathbf{w} \rangle$
4. $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v} + \mathbf{w} \rangle$
5. $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u} + k\mathbf{v} \rangle$

Exercise: Proof the theorem!

Weighted inner product

If $w_1, w_2, \ldots, w_n \in \mathbb{R}$ and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$. Then:

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

is called weighted Euclidean inner product with weights $w_1, w_2, \ldots, w_n$.

Example

Exc 1: Calculation with weighted Euclidean inner product

The standard inner product on M_{nn}

Given $\mathbf{u} = U$ and $\mathbf{v} = V$, matrices in the vector space M_{nn}. The standard inner product on M_{nn} is defined as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{tr}(U^T V)$$

Example

For $\mathbb{R}^2$, and matrices:

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

The standard inner product of U and V is:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Exercise: Prove that the four axioms of inner product hold!

The evaluation inner product on P_n

Let $\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$ and $\mathbf{q} = b_0 + b_1 x + \cdots + b_n x^n$ are polynomials in P_n.
$x_0, x_1, \ldots, x_n$ are distinct real numbers.

The evaluation inner product on P_n at $x_0, x_1, \ldots, x_n$ is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n)$$

Exercise: Prove that the four axioms of inner product hold!

Exc 2: Calculation with a weighted Euclidean inner product

Part 2: Angle and orthogonality in inner product spaces

Norm of vector and the algebraic properties

Let V be a real inner product space. Then: the norm of $\mathbf{v} \in V$:

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Theorem (Properties of norm)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V*, and if* $k \in \mathbb{R}$*, then:*

1. $\|\mathbf{v}\| \geq 0$ *where equality holds iff* $\mathbf{v} = \mathbf{0}$*;*
2. $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$*.*

Exercise: Proof the theorem!

Distance of two vectors and the algebraic properties

Let V be a real inner product space. Then: the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$:

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle}$$

Question: can you relate the above definition with the definition of distance that we discussed earlier?

Theorem (Properties of distance)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V*, and if* $k \in \mathbb{R}$*, then:*

1. $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$;
2. $d(\mathbf{u},\mathbf{v}) \geq 0$ *where equality holds iff* $\mathbf{u} = \mathbf{v}$.

Exercise: Proof the theorem!

Angle between two vectors in $\mathbb{R}^n$

Recall that:

The angle between two vertices **u** and **v** in $\mathbb{R}^n$ is defined as:

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

Theorem (Cauchy-Schwarz inequality)

If **u** *and* **v** *are vectors in a real inner product space V, then:*

$$|\langle\mathbf{u},\mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

Note that the theorem implies that the angle between **u** and **v** ranges between 0 and $\pi = 180^o$.

$$0 \leq \theta \leq \pi$$

Exercise: Prove the Cauchy-Schwarz inequality!

Angle between two vectors in real vector space

The Cauchy-Schwarz inequality implies:

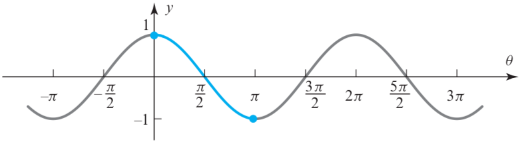
$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1$$

which means that there is a <u>unique angle θ</u> for which:

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi$$

Hence,

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)$$



Angle of vectors in M_{22}

In the previous example, we are given two vectors in M_{22}, namely $\mathbf{u} = U$ and $\mathbf{v} = V$, where:

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

How to find the angle between $\mathbf{u}$ and $\mathbf{v}$?

Solution:

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

- Compute $\langle \mathbf{u}, \mathbf{v} \rangle$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1(-1) + 2(0) + 3(3) + 4(2) = -1 + 0 + 9 + 8 = 16$$

What can you conclude about the angle of two vectors in an inner product space?

Triangular inequalities

Theorem (Triangular inequalities on norm and distance)

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in a real inner product space* V, *and* $k \in \mathbb{R}$, *then:*

1. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
2. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

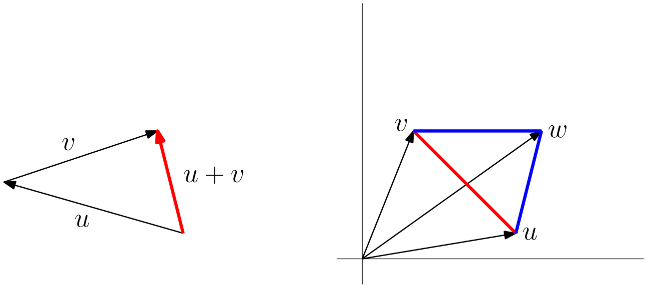


Figure: Illustration of triangular inequality

Orthogonality

Definition

Two vectors **u** and **v** in an inner product space V called orthogonal if $\langle u, v \rangle = 0$.

Example 1: Orthogonality depends on the inner product

Given two vectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$.

- $\mathbf{u}$ and $\mathbf{v}$ are orthogonal w.r.t. the Euclidean inner product on $\mathbb{R}^2$.

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (1)(-1) = 0$$

- but not orthogonal w.r.t. the weighted Euclidean inner product: $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$, since:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0$$

Example 2: Orthogonal vectors in M_{22}

Recall the definition of the inner product on M_{nn} :

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{tr}(U^T V)$$

Are the following matrices orthogonal?

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution:

$$\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$$

Hence, U and V are orthogonal.

Example 3: Orthogonal vectors in P_2

Define inner product on P_2 as follows.

For $\mathbf{p}, \mathbf{q} \in P_2$, define:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\ dx$$

Let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then:

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left[\int_{-1}^{1} xx\ dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\ dx\right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left[\int_{-1}^{1} x^2x^2\ dx\right]^{1/2} = \left[\int_{-1}^{1} x^4\ dx\right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2\ dx = \int_{-1}^{1} x^3\ dx = 0$$

Hence, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the given inner product.

Exercise

Prove the following Generalized Theorem of Pythagoras

If **u** and **v** are orthogonal vectors in a real inner product space, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

to be continued...